Recitation 9/26 Recitation Solutions

1. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 4 \end{bmatrix}$.

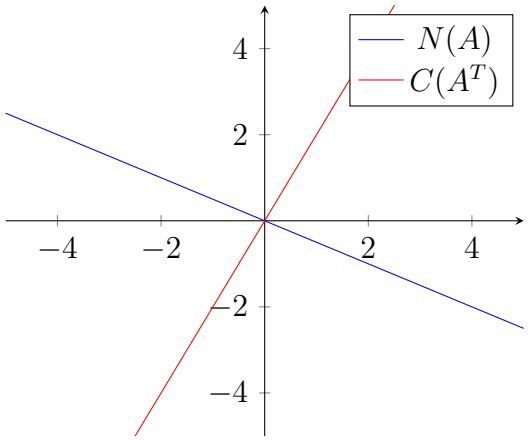
   (a) Find bases for $N(A)$ and $C(A^T)$ and sketch these subspaces together in $\mathbb{R}^2$.

   The RREF of A is: $\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

   Solving $A\mathbf{x} = \mathbf{0}$ corresponds to solving the system $x_1 + 2x_2 = 0$. The general solution to this system is $\begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Hence $N(A) = \text{Span}\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}$. It follows that $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ is a basis for $N(A)$.

   Since the pivot rows of A form a basis of the row space of A and the first row is the only pivot row, we have that $C(A^T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and so $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $C(A^T)$.

   $N(A)$ is given by the line $y = -\frac{1}{2}x$ and $C(A^T)$ is given by the line $y = 2x$, sketched below.



   (b) Find bases for $C(A)$ and $N(A^T)$ and sketch these subspaces together in $\mathbb{R}^3$ (you can try to do this by hand or visit `https://www.geogebra.org/m/t4cwdjqa`).

   Since the pivot columns of A form a basis of $C(A)$, we have that $C(A)$ has basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ and $C(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

To find a basis for $N(A^T)$. We need to solve the system $A^T\mathbf{x} = \mathbf{0}$. The RREF of A^T is $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Solving $A^T\mathbf{x} = \mathbf{0}$ corresponds to solving the system $x_1 - x_2 + 2x_3 = 0$, where x_2 and x_3 are free. The general solution to this system is $\begin{bmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Hence $N(A^T) = \text{Span}\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \}$. It follows that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $N(A)$.

Visit the link above to see $N(A^T)$ represented by a plane and $C(A)$ represented by a line in $\mathbb{R}^3$.

2. Let A be an $m \times n$ matrix. What can you say about m and n if:

   (a) $C(A) = \mathbb{R}^m$?

   If $C(A) = \mathbb{R}^m$, then the columns of A span $\mathbb{R}^m$. If $n < m$ (there are more rows than columns), then there cannot be a pivot in every row of A. As a result, there would be a vector $\mathbf{b} \in \mathbb{R}^m$ such that the augmented matrix $\begin{bmatrix} A & | & \mathbf{b} \end{bmatrix}$ has a pivot in the final column and so $A\mathbf{x} = \mathbf{b}$ has no solution; this implies that the columns of A do not span $\mathbb{R}^m$. Hence we must have that $n \geq m$.

   (b) $N(A) = \{\mathbf{0}\}$?

   If $N(A) = \{\mathbf{0}\}$, then the system $A\mathbf{x} = \mathbf{0}$ only has the zero solution. This implies that there are no free variables and so A must have a pivot in every columns of A. This can only happen if there are at least as many rows as columns; that is $m \geq n$.

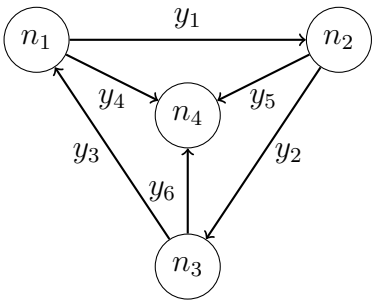
   (c) $N(A) = N(A^T) = \{\mathbf{0}\}$?

   If $N(A) = N(A^T) = \{\mathbf{0}\}$, then $\dim N(A) = \dim N(A^T) = 0$. Since $\dim N(A) = n - r$ and $\dim N(A^T) = m - r$, we have that $n - r = 0$ and $m - r = 0$; hence $n = m = r$.

   (d) $C(A) = \mathbb{R}^m$ and $C(A^T) = \mathbb{R}^n$?

   If $C(A) = \mathbb{R}^m$ and $C(A^T) = \mathbb{R}^n$, then $\dim C(A) = m$ and $\dim C(A^T) = n$. But now since $\dim C(A^T) = \dim C(A)$, we have that $n = m$.

3. **Left Nullspace Application:** Consider the graph below consisting of edges and nodes.



Suppose this is an electrical network and y_i represents the current traveling along each edge as labeled above; the arrows on the edges indicate the direction of the current **Kirchhoff's Current Law** says that "current in = current out" at every node. For example, at node n_i, we have $y_3 = y_1 + y_4$. We can summarize this network using an **incidence matrix** defined as follows:

- The columns correspond to the nodes
- The rows correspond to the edges
- The entries are $a_{ij} = \begin{cases} -1 & \text{if edge } y_i \text{ leaves node } n_j \\ 1 & \text{if edge } y_i \text{ enters node } n_j \\ 0 & \text{otherwise} \end{cases}$

(a) Write down the incidence matrix for the graph above.

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(b) Write the four equations for Kirchhoff's Law at the four nodes.

The equations are (in order of nodes)

$$\begin{aligned} y_3 &= y_1 + y_4 \\ y_1 &= y_2 + y_5 \\ y_2 &= y_3 + y_6 \\ y_4 + y_5 + y_6 &= 0 \end{aligned}$$

(c) Show that solutions to the above system correspond to solutions to $A^T\mathbf{y} = \mathbf{0}$ (i.e. the left nullspace of A).

Rewriting the above system gives:

$$\begin{aligned} -y_1 + y_3 - y_4 &= 0 \\ y_1 - y_2 - y_5 &= 0 \\ y_2 - y_3 - y_6 &= 0 \\ y_4 + y_5 + y_6 &= 0 \end{aligned}$$

Rewriting this system as a matrix equation gives:

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The coefficient matrix above is exactly A^T. Hence the solution set to Kirchhoff's equations is precisely the left nullspace $N(A^T)$.

(d) Find a basis for $N(A^T)$.

The RREF of A^T is $\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

This corresponds to the system

$$\begin{aligned} y_1 - y_3 - y_5 - y_6 &= 0 \\ y_2 - y_3 - y_6 &= 0 \\ y_4 + y_5 + y_6 &= 0 \end{aligned}$$

where y_3, y_5 and y_6 are the free variables. This gives rise to the general solution

$$\mathbf{y} = \begin{bmatrix} y_3 + y_5 + y_6 \\ y_3 + y_6 \\ y_3 \\ -y_5 - y_6 \\ y_5 \\ y_6 \end{bmatrix} = y_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + y_6 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $N(A^T)$.

(e) Optional: Interpret each vector in the basis in terms of cycles in the network. Explain why any other cycle in the network can be built by combining cycles in this basis.

We interpret a vector $\mathbf{y}$ as encoding the amount of current flowing through each edge. In this interpretation, $A^T\mathbf{y} = \mathbf{0}$ forces each node to have the same amount of current flowing in as flowing out. Hence a vector $\mathbf{y}$ is in the nullspace if and only if this vector has the net amount of current into each node equal to 0. For example, the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ corresponds to there being 1 unit of current through the edges y_1, y_2, and y_3; notice that at each node, the net current is 0. Note the three basis vectors found in the previous part correspond to sending a unit amount of current around each of the following three cycles in the graph (+1 if the current flows in the direction of the arrow, -1 if it flows the other way):

- y_1, y_2, y_3
- y_1, y_5, y_4
- y_1, y_2, y_6, y_4

Notice that there are other cycles we could send current through that do not correspond to the basis vectors. The fact that the three cycles above correspond to basis elements tells us that all other net zero flows are linear combinations of these three. For example the flow sending 1 unit of current around the cycle y_3, y_4, y_6 is given by subtracting the third cycle from the first. A set of cycles with this property is known as a **cycle basis** for a network.